Linear Algebra and Multivariable Calculus for Chemistry
2016/17 Semester Ib
Instructor: Daniel Valesin
Final Exam
24/1/2017
Duration: 3 hours

Name: ____________________

Student number: __________

This exam contains 8 pages (including this cover page) and 7 problems.

Enter all requested information on the top of this page.

Your answers should be contained **in this exam booklet**. Avoid handing in extra pages.

You are required to show your work on each problem.

Do not write on the table below.

Problem	Points	Score
1	15	
2	11	
3	11	
4	10	
5	15	
6	14	
7	14	
Total:	90	

(+10 free points)

1. Let $\vec{u} = (-8, 10, 32)$. For t in $\mathbb{R}$, let

$$A_t = \begin{pmatrix} 3 & -1 & -1 \\ 0 & t & 2 \\ -12 & 4 & t \end{pmatrix}.$$

   (a) (5 points) Find $\det(A_t)$ as a function of t.

   (b) (5 points) Letting $t = 4$, find the solution set of $A_t\vec{x} = \vec{u}$.

   (c) (5 points) For what values of t does $A_t\vec{x} = \vec{u}$ have: exactly one solution, no solution, infinitely many solutions?

2. (a) (5 points) The linear transformations $S, T, U : \mathbb{R}^2 \to \mathbb{R}^2$ act on vectors as follows:

   - $S(\vec{x})$ is the reflection of $\vec{x}$ about the line $x + y = 0$;
   - $T(\vec{x})$ is the result of rotating $\vec{x}$ counterclockwise with angle $\pi/4$;
   - $U(\vec{x})$ is the projection of $\vec{x}$ into the y axis.

   Find the matrix of the transformation $R(\vec{x}) = U(T(S(\vec{x})))$.

   (b) (6 points) Determine the set

$$A = \{(x, y) \text{ in } \mathbb{R}^2 \text{ such that } R(x, y) = (0, 1)\}.$$

   Draw a graph sketching A and the sets

$$B = \{S(x, y) : \ (x, y) \text{ belongs to } A\},$$
$$C = \{T(S(x, y)) : \ (x, y) \text{ belongs to } A\}.$$

3. The **trace** of a square matrix A, denoted trace(A), is the sum of the entries in the diagonal of A. The trace is known to have the following property: if A and B are similar matrices, then they have the same trace.

   (a) (6 points) Verify that this property indeed holds true for the matrices A and $B = PAP^{-1}$, where

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 4 \\ 9 & 8 & 7 \end{pmatrix}, \qquad P = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix}.$$

   (b) (5 points) Assume Q is a 3×3 diagonalizable matrix with eigenvalues 1, 2 and 3. What is the trace of Q?

4. (10 points) Find the solution of the linear differential equation

$$\vec{x}'(t) = \begin{pmatrix} 5 & 0 & 6 \\ 0 & 2 & 0 \\ 3 & 0 & 2 \end{pmatrix} \vec{x}(t)$$

with the initial condition $\vec{x}(0) = (1, 1, -1)$.
Hint. When looking for eigenvalues, remember that a determinant can be found by cofactor expansion across any row or column we choose.

5. Consider the function

$$f(x, y, z) = x^3 yz - \frac{yz}{x^2 + 1}.$$

(a) (5 points) Find $\nabla f(x, y, z)$.

(b) (5 points) Find the directional derivative of f at the point $(x, y, z) = (2, 0, 3)$ and in the direction $\vec{u} = \frac{1}{\sqrt{6}}(1, 2, -1)$.

(c) (5 points) Find the equation of the tangent plane to the level surface of f at the point $(x, y, z) = (2, 0, 3)$.

6. (a) (7 points) Assume $m > 0$ and $f : \mathbb{R}^n \to \mathbb{R}$ is a function satisfying

$$f(t\vec{x}) = t^m \cdot f(\vec{x}) \quad \text{for all } t \geq 0,\ \vec{x} \text{ in } \mathbb{R}^n.$$

Show that

$$\nabla f(\vec{x}) \cdot \vec{x} = m f(\vec{x}) \quad \text{for all } \vec{x} \text{ in } \mathbb{R}^n.$$

Hint. For fixed $\vec{x}$ in $\mathbb{R}^n$, define $g(t) = f(t\vec{x})$ and compute $g'(1)$.

(b) (7 points) Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$ be differentiable functions. Assume that

$$f(5) = 2, \quad f'(5) = -3, \quad \nabla g(x, y) = \left(-\frac{y^2}{(x+1)^2}, \frac{2y}{x+1}\right).$$

For $h(t) = g(f(t), 1/f(t))$, find $h'(5)$.

7. (a) (7 points) Sketch the region of integration and invert the order of integration:

$$\int_2^4 \int_0^{(x-2)^3} f(x,y)\, dydx.$$

(b) (7 points) Evaluate the integral:

$$\int_0^{\sqrt{\pi}} \int_y^{\sqrt{\pi}} \sin(x^2)\, dxdy.$$